A NEW THEORY OF MAGNETIC STORMS*
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PART I—THE INITIAL PHASE (Continued)

6—The phenomena accompanying the advance of a solar ionized stream into the Earth's field.

6.1—A stream of particles emitted nearly radially from an active area A on the Sun will have a curved form, on account of the Sun's rotation. The stream will lag slightly behind the solar radius through the emitting area A, though each particle of the stream will move almost rectilinearly, approximately along the solar radius through the position of A at the instant of emitting that particle. The rate at which the lateral surface of the stream will approach the Earth, overtaking it in its orbital motion, depends only on the Sun's known angular velocity; the point of intersection of the Earth's orbit with the lateral surface of the stream moves relative to the Earth with a velocity of about 400 km/sec, thus traversing a distance equal to one Earth-radius in about 18 seconds. If undisturbed by the Earth's field the stream would sweep across the Earth in about 35 seconds.

The advancing surface of the stream would take about 15 minutes to reach the Earth from a distance of 50 Earth-radii, where the Earth's magnetic intensity is only 0.3\gamma. After a further 15 minutes the forward surface would have passed through the more intense part of the field to the distance at which the intensity is once more only 0.3\gamma. From this time until 15 minutes before the following surface of the stream sweeps across the Earth, the whole of the region in which the magnetic field has an intensity exceeding 0.3\gamma is within the stream. If this period is assumed to be of the same order of duration as the active period of a magnetic storm, it is about one day. This would correspond to a diameter of cross-section of the stream, at the Earth's orbit, of about 5,000 Earth-radii.

If this be the order of magnitude of the diameter of the stream, then as viewed from the Earth the stream will subtend an angle of nearly 2\pi (the angle subtended at any point by an infinite plane) when the nearest point of the stream is at a distance of 50 radii from the Earth (assuming that at this distance the surface is undistorted by the field). Actually, at this distance, the angle subtended by the stream (which can for this purpose be treated as an infinite cylinder of diameter 5,000 Earth-radii) is about \gamma/2 of 2\pi. The radius of curvature of the surface being fifty times the distance from the Earth, the surface will appear nearly plane as viewed from the Earth.

6.2 In the earlier stages of our work we made many attempts to find whether the charged surface-layer on the stream, due to the polarization by the magnetic field, could produce any appreciable magnetic effect near the Earth, either during the approach of the stream-boundary towards the Earth, or while the Earth is enveloped in the stream. One great difficulty in making these attempts was the essentially three-dimensional character of the problem; it seemed scarcely possible to seek the usual refuge of the applied mathematician in analytical distress, and attempt as an alternative to solve some sufficiently illustrative

*Continued from this JOURNAL, 36, 77-97 (1931). The following correction is to be noted on page 83 in the sixth line of the last paragraph of Section 1: For "in Figure 1" read "in Figure 5."
analogous problem in two dimensions; two-dimensionality seemed attainable only by omitting some essential feature of the phenomena. It was at this stage that, starting with the problem of §5, we were forced back to examine the simpler problems of §§2-4. Also other simple problems were attempted, and partly solved, concerning the state of streams of small section approaching or passing by the Earth. It is unnecessary to describe the details of these attempts; their result was to convince us that, as in the simple cases of §§2 and 3, the magnetic field, near the Earth, due to the moving charged layers, regarded as electric currents, was negligible.

6.3—We also endeavoured to find whether the charges escaping from the surface, under the influence of the external electric field of the charged layer, could provide sufficient current, in the Earth's neighbourhood, to produce an appreciable disturbance of the magnetic field; the surface-charge on the stream, when approaching the Earth, is negative, and the escaping particles are electrons. Our conclusion, after detailed examination, was that their terrestrial magnetic effect was negligible; when the stream is many Earth-radii distant from the Earth, the surface-density of charge, the external electric field, and the rate of escape of charge, are all very small; moreover the escaping electrons do not approach near the Earth, at least in the equatorial plane, though many of them may find their way along the Earth's lines of force towards the auroral zones.

The "escape" of particles from the surface, at this distance, is analogous to the very large lateral motion (§2.7) of the surface-layer of an infinite plane-slab stream in a very weak uniform field; in the terrestrial case, owing to the non-uniformity of the field outside the stream, and its great increase of intensity around the Earth, the motion of the charged layer of the stream will not be simply periodic and pulsatory; the charges may pass quite beyond the reach of the electric field of the stream, and never return to it. But as the stream approaches the Earth, the "spiral radius" of the electrons in the charged layer will be progressively reduced; for example, at a distance of 50 Earth-radii, or 15 minutes travel of the surface, from the Earth, the spiral radius of the outermost electrons, at the point on the surface of the stream that is nearest to the Earth, will be only about 20 km, and in the equatorial plane the electrons will not "escape" from the surface further than this, but will oscillate to and from the surface over this distance. Those not in the equatorial plane can move farther away, and may escape towards the auroral zone along the Earth's lines of force. But since the increasing negative charge of the surface-layer, as it approaches the Earth, coincides with increasing attachment of the layer (by the influence of the Earth's magnetic field) to the surface, the equivalent current, and its magnetic field, will approximate more and more in order of magnitude (or upper limit) to those that can be estimated by analogy with §2.4; \( H' \) will be altered at most by an amount of the order \( (V^2/c^2)H' \), where \( H' \) is the intensity of the Earth's field at the layer; if \( V = 10^8 \), this would be negligible even if the layer were in contact with the Earth. If \( V = 10^9 \), \( (V^2/c^2)H' \) would not then be negligible: but we shall show that, actually, the surface cannot approach within several Earth-radii before the field greatly distorts it, and induces other phenomena of much greater importance than the magnetic effect of the (undistorted) charged surface-layer. We therefore attribute no importance, for the theory of magnetic
storms. to the surface-charge on the boundary of the solar stream, while this is approaching the Earth with little or no distortion.

6.4—We believe our first real insight into the mode of production of magnetic storms was gained when we perceived that a further surface-effect would arise owing to the advance of the stream into the Earth's field—an effect not paralleled in the illustrative problems previously considered. The stream, as mentioned in the introduction (§1.6), is an electrical conductor, and the motion of the terrestrial magnet relative to this conductor will induce electric currents in it, tending to shield the interior of the stream from the changes in the field. These currents initially flow near the surface, and parallel to it; gradually they diffuse inwards, and the degree of magnetic shielding in the regions newly invaded by the currents is progressively reduced.

If the conductivity of the stream, and its rate of advance into the Earth's field, are sufficiently great, so that the stream is a nearly perfect conductor, the interior will be almost completely shielded from the Earth's field. The tubes of force occupying the region traversed by the field will be, as it were, pushed forward by the stream. This will increase the magnetic intensity around the Earth—an effect which is characteristic of the first phase of a magnetic storm.

6.5—This compression of the Earth's tubes of magnetic force into a smaller volume represents an increase in the magnetic energy of the field, gained at the expense of the kinetic energy of the stream, whose particles are retarded by the field. The retardation here mentioned is additional to, and greater than, the one previously discussed; that was due to the polarization of the stream by the field, and the lost kinetic energy was converted partly into electrostatic energy and partly into magnetic energy, though the change in the latter was very small; the whole body of the stream is subject to the retardation due to the polarization. The retardation of the stream due to the magnetic shielding of its interior is effected by electromagnetic forces acting on the induced currents flowing near the surface; the retarding force acts only on the part of the stream, near the surface, which conveys the electric currents, while the interior is unaffected. It is, of course, true that the shielding of the interior of the stream from the magnetic field removes the polarizing forces on the interior, and so eliminates the former type of retardation.

6.6—The retarding force per unit surface-area of this current-layer at any point is proportional to the tangential intensity \( I_s \) of the field, and to the current density \( i \) at the point; and \( i \) will be approximately proportional to the normal velocity \( v_n \) of the surface, and to \( I_s \). In all, the retarding force will be approximately proportional to \( v_n I_s^2 \). In the equatorial plane \( H_s = H \), the surface being there, by symmetry, perpendicular to this plane; since \( H \propto 1/r^2 \), where \( r \) denotes distance from the Earth's centre, the retarding force varies as \( v_n/\rho \); this rapidly increases as the surface approaches the Earth, except in so far as the velocity of approach, \( v_n \), is simultaneously reduced.

The stream will first be retarded, during its approach to the Earth, on the advancing side, nearest the Earth. As the stream continues to advance, the approaching particles near the surface on the sunward side of the Earth will be increasingly retarded, and the surface will become bent round the space occupied by the Earth's magnetic field. After the time when, if the field had been absent, the surface of the stream would
have swept past the Earth, the actual surface, on the forward side, will be entering into a weaker field, where it is less retarded; thus it will be able to advance, along the direction of motion of the particles, beyond the part of the surface immediately between the Sun and the Earth. A hollow space, of which the equatorial section is roughly parabolic in form, with its axis lying nearly along the line from the Sun to the Earth, and its vertex on this line (not far from the Earth on its sunward side), will thus be formed by distortion of the stream-surface. Successive stages in the formation of this hollow are shown in Figure 2 by sections in the equatorial plane.

6.7.—In so far as the interior of the stream is shielded from the magnetic field, the ions and electrons there will move forward without deflection or retardation due to polarization. Those immediately behind the current-bearing surface-layer will therefore overtake this retarded layer, and increase its mass-density; they will then share in the transmission of the induced current, and in the consequent retardation; the effect of the increased density upon the electric conductivity is discussed in §8.

The rate of retardation of the surface is governed by the magnitude
of the retarding force, and the mass on which it acts. The latter con-
stantly increases by the inpouring of particles from the interior into the
current-bearing layer, and also by the gradual backward diffusion of the
currents, and the consequent thickening of the layer. The retarding
force (roughly proportional to \( v_m/r^6 \)) tends to increase on account of the
increased intensity of the field into which the surface advances; but the
decreasing velocity of advance tends to reduce the force. The surface
will continue to advance, at a decreasing rate; the actual rate of retarda-
tion, and the effect on the magnetic field, are matters for detailed calcu-
lation (§7). They depend essentially on the momentum of the stream
per unit volume. This is probably sufficiently small (§1.6) for the
velocity of the surface to be greatly reduced.

6.8—If there were no upper limit to the density of the streams which
the Sun could emit, magnetic storms might be produced that would
have characteristics markedly different from those that are observed;
very dense streams would be only slightly retarded, and though they
would produce an increase in the field near the Earth during their
approach, this would be a fleeting phenomenon; the "hollow" would in
their case be only a slight depression in the advancing surface, which
would sweep up to and across the Earth; only the region in the shadow
of the Earth would then be free from the stream.

On the other hand, there is a lower limit to the stream-density that
suffices for the production of the phenomena described in this section
(§6); the ions and electrons of a very rare stream would be separately
deflected as if they were independent particles, and the stream could
not be regarded as a conducting medium. In a very rare stream of ions
and electrons the advance of the electrons would be stopped at a far
greater distance from the Earth than that to which the ions could attain;
in this case the excluded spaces inaccessible to the particles of either kind
are those discussed in Störmer's auroral theory.

6.9—The analytical discussion of the distortion of the surface of the
stream by the Earth's field, and the magnetic consequences, offer diffi-
culties which we have not yet overcome or, indeed, seriously attacked
as yet. Our course has been to consider simpler problems that seem to
illustrate the main features of the actual case with sufficient closeness
to enable us to draw numerical conclusions of at least the right order of
magnitude.

We begin by considering the current-system in a rigid infinite plane
conducting-sheet approaching the Earth. The bearing of the results in
this case, upon the problem of the non-rigid conducting-stream, is then
discussed. This is followed by a numerical calculation intended to
illustrate the rate of advance of the apex of the hollow surface, as a
function of the momentum-density of the stream. The increase of
magnetic intensity within the hollow, and its rate of change, are estimated,
and it is suggested that they can reasonably be identified with those that
are observed in the first phase of a magnetic storm. The identification
leads to numerical estimates of the momentum-density of the solar
streams.

7—The currents induced in a conductor moving in a magnetic field,
and their magnetic effects

7.1—We first consider the currents induced in an infinite thin plane
conducting-sheet \((s=0)\), by a magnetic system of any kind (on the side
z > 0) advancing with constant velocity, \(-w\), normal to the sheet. Maxwell\(^3\) showed that the problem could be reduced to the determination of a single function, so long as the ethereal displacement-currents are negligible, which is the case if \(w/c\) is small. We denote the current (or stream) function for the sheet by \(\Psi\), the magnetic potential of the inducing system by \(\Omega\), the magnetic potential due to the current-sheet, in the region on the same side as the inducing magnetic system, by \(\Omega'\), and the specific resistance per unit area of the sheet by \(2\pi w_0\); \(w_0\) has the dimensions of a velocity. It is clear that \(\Omega'\) will be an even function of \(z\), the field of the current-system being symmetrical with respect to the sheet.

The equations of induction being linear, the effects of different systems are additive, and so it is sufficient to consider the simple case of a magnetic dipole, taken to be of moment \(M\), at a point \(P\) \((z > 0)\). Let \(P'\) be the image-point of \(P\) with respect to the sheet, that is, the point on the same normal, at an equal distance from the sheet on the side \(z < 0\). Then the magnetic effect of the currents induced in the sheet is to reduce the field on the further side \((z < 0)\) in the ratio \(w_0/(w + w_0)\), the same at every point, as if the sheet were absent and the doublet at \(P\) were of moment \(Mw_0/(w + w_0)\). On the side \(z < 0\) an additional field, such as would be produced in that region, in the absence of the sheet, by a positive image-doublet of moment \(Mw/(w + w_0)\) at \(P'\), is superposed on the field of the doublet \(M\); this increases the magnetic energy of the field in this region, though the intensity is not changed everywhere in the same ratio, nor even everywhere increased.

These results are valid whatever the direction of the doublet. They are unaffected also if the sheet moves towards the doublet instead of vice versa, the relative motion being the determining factor (the rate of propagation of changes in the field is here regarded as infinite).

Any magnetic system can be built up by superposition of simple doublets. Hence the above discussion indicates that on the further side of the sheet the field of the system, when moving as a whole towards the sheet, at right-angles, is reduced in the ratio \(w_0/(w + w_0)\). The change in the field on the same side as the system itself corresponds to the superposition of a field of the same character as the unaltered field of the system on the further side, but of only \(w/(w + w_0)\) times the intensity.

7.2—The case of most interest for our problem is that in which the doublet is parallel to the sheet. If its axis is taken to be in the \(y\)-direction, its potential at \((x, y, z)\), when \(P\) is at the point \((0, 0, c)\), is \(My/r^3\), where

\[
(23) \quad r^2 = x^2 + y^2 + (z - c)^2
\]

The current-function \(\Psi\) is given by

\[
(24) \quad \Psi = \frac{w}{w + w_0} \frac{My}{2\pi r_0^3}
\]

where \(r_0^2 = x^2 + y^2\). Thus the current-lines are the lines of intersection of the sheet with the equipotential surfaces of the field; they are shown in Figure 3. The current-system consists of two families of ovals, separated by the line \(y = 0\); their foci or limiting points \(Q\) are at \((0, \pm c/\sqrt{2}, 0)\), where the equipotential surfaces touch the sheet, that is, where

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\(^3\)J. C. Maxwell, Electricity and magnetism, 2d ed., Oxford, 1881, Ch. XII, § 657.
the magnetic lines of force of the doublet cross the sheet at right-angles. At any other point the current-components are

$$\frac{1}{2\pi} \frac{w}{w+w_0} (-H_y, H_x, 0)$$

where $H_x, H_y$ are the $x, y$ components of the field of the doublet. In

Fug. 3

the equatorial plane of the doublet, $y=0$, the current is along the $x$-direction, and of amount

$$\frac{1}{2\pi} \frac{w}{w+w_0} \frac{M}{(x^2+c^2)^{3/2}}$$

If the sheet were a perfect conductor ($w_0=0$) the far side would be completely shielded from the field of the doublet, and the image-doublet which gives the additional field on the near side would have the full moment $M$.

7.3—The determination of the induced currents and magnetic effect is more difficult when the magnetic system moves parallel to the sheet. If the velocity $u$ parallel to the sheet is large compared with $w_0$, the field of the current-sheet would almost annul the field on the far side, and on the near side would add to the field by a corresponding amount; for example, for a doublet $M$, by nearly the field of the full image-doublet.
but in addition there is a small lagging field (on both sides) of order \( \omega_0/u \) times the field just described.

The problem of a magnetic system moving parallel to and between a pair of highly-conducting infinite thin parallel plane-sheets can be treated approximately by the method of images. It is easy to see that, provided the magnetic system is not too near either sheet, the magnetic field between them will on the whole be increased by about twice or thrice the amount which either sheet alone could produce.

7.4—If the conductor is not a thin plane-sheet, but a body (of specific resistance \( 2\pi\rho_0' \)) extending throughout the whole space on the negative side of the plane boundary \( z = 0 \), the induced currents will flow parallel to the surface, their intensity decreasing with increasing depth from the surface. The currents would be unaffected if the body were divided into a series of thin contiguous sheets parallel to the surface; each sheet would be subject to a magnetic field similar in character to the one that would exist there if the sheets between it and the surface were removed, but reduced in intensity by the shielding effect of the currents in these sheets. The magnetic field would thus be progressively reduced inside the body; the ratio of reduction at not too great a depth \( d \) below the surface would be approximately the same as that due to a thin sheet of specific resistance \( 2\pi\rho_0'd \), placed somewhere between \( z = 0 \) and \( z = d \).

The field outside the conductor would no longer be modified by the addition merely of a field equivalent to that of a single image-magnet; but if the currents are concentrated in a layer of which the thickness is small compared with the distance from the actual magnet to the surface of the conductor, their external field is nearly equivalent to that of an image-doublet in a perfectly conducting thin plane-sheet.

7.5—When the boundary of the body is not plane, the determination of the induced currents and their magnetic field is more difficult. Larmor showed in 1884 that in the case of a conducting sphere, or spherical shell of sufficient thickness, "no external magnetic disturbance whatever can induce currents which do not circulate in concentric shells." Substantially the same conclusion may be drawn whatever the form of the surface, if its curvature changes only gradually from point to point; the penetration of a conductor by a varying magnetic field will be reduced, and, if the conductivity be sufficient, will be confined to a thin surface-layer, flowing in this layer nearly parallel to the surface; the current-intensity will decrease to a low value in a stratum of thickness varying inversely as the specific resistance \( \omega_0' \) of the body. If the body is of small dimensions, the available total conductivity may be inadequate to shield the interior to any appreciable extent, but whether the interior is well shielded or not, the external field of the induced currents will be small, because the displacement of the tubes of force which would otherwise pass through the body affects only a correspondingly small region of the field.

In all cases of complete shielding of a magnetic field from the region bounded by a perfectly conducting sheet of infinitesimal (or small) thickness, the added field outside the surface must be such as to destroy the normal component of the resultant field \( \mathbf{H} \) at the surface: this is obvious because otherwise magnetic tubes of force would enter the shielded region. Figure 4 gives a rough indication of the combined

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\(^{2}\)Mathematical and physical papers, Cambridge, 1939, 1, p. 13.
field in the case of a doublet parallel to, and approaching, a perfectly conducting thin plane-sheet; the position of the image-doublet is indicated on the right. When the shielding is only partial, the inclination of the tubes of force to the surface is reduced, though not annulled, by the induced currents, so that fewer tubes enter the surface.

7.6—We next consider the mechanical forces arising between the magnetic system and the conductor in whose neighbourhood the magnetic system is moving. One case can be dealt with specially simply, without considering the detailed interaction between the magnetic field and the surface-currents.

Consider a magnetic doublet of moment $M$, with its axis parallel to the plane-face of a semi-infinite conductor at distance $r$. When the doublet is approaching the conductor at a rate sufficiently rapid (in relation to the conductivity of the solid) to cause complete shielding within a thin surface-layer, the currents produce a magnetic field, in external space, equivalent to that of a doublet of equal moment $M$, at a distance $2r$ from the real magnet. The resultant mechanical force $F$ between the magnet and the solid is the same as that between the real doublet and the image-doublet, that is

$$F = \frac{3M^2}{(2r)^4}$$
The work done in bringing the doublet (or the solid) from infinity to the distance $r$ is $\int_{\infty}^{r} F \, dr$, or

\begin{equation}
M^2/16r^3
\end{equation}

If $r=a$, where $a$ is the Earth's radius, the work done is

\begin{equation}
M^2/16a^3
\end{equation}

or, inserting numerical values for $M$ and $a$, it is

\begin{equation}
1.7 \times 10^{-4}/r^3
\end{equation}

since $M=8.5 \times 10^{20}$, $a=6.37 \times 10^8$.

7.7.—The mechanical force on an element of a conductor carrying an electric current of density $i$ is $i \times H$ per unit volume, where $H$ is the magnetic intensity at the point, and $i$, $H$ are measured in electromagnetic units. For a thin rigid infinite perfectly conducting plane-sheet, the mechanical force per unit area is $i \times H$, where $i$ is the current-density in two dimensions; in the case considered in §7.2, this varies from point to point, being a maximum at the point $(0, 0, 0)$, and zero at the limiting points $Q$.

Suppose the doublet to be at rest, and the thin sheet in normal motion towards it; if the sheet suddenly ceased to be rigid, distortion of the plane would occur. The momentum would initially be the same per unit area all over the plane, but the retardation would vary from point to point, tending to zero at infinity; hence the outlying parts of the sheet would advance relatively to the parts around the origin. The parts near the foci $Q$ of the current-ovals would also advance relatively to $O$ and to other points where the current-density is comparable with that at $O$. Thus the sheet would tend to be held back in the regions round the origin, except for two "horns" that would tend to form round the points $Q$. As the sheet approached the Earth, the foci $Q$ of the current-ovals would, if the sheet remained plane, move along a straight line towards the Earth's centre, their mutual distance therefore steadily decreasing. It seems likely that the distance between the horns will likewise lessen, although the sheet becomes distorted as it approaches the Earth. This progressive change of location of the foci $Q$ on the advancing stream-front may somewhat restrict the passage of matter from the rear, through these patches of little or no retardation, into the space on the earthward side of the front; but such passage seems bound to occur. The matter thus passing through seems likely to find its way ultimately towards the polar regions, but a detailed discussion of this point is deferred.

These horns complicate the form of the section of the front of a solar stream in the terrestrial-meridian plane through the Sun, but nevertheless the section in this plane must become curved so as to enclose the Earth. The section by the equatorial plane (Fig. 5) will be simpler, and of a roughly parabolic form, the less-retarded sides advancing relatively to the more-retarded centre.

Owing to this distortion of the stream-front, the space shielded from the Earth's field is greater, for a given distance of the centre $O$ of the front from the centre $C$ of the Earth, than if the front remained plane. The greater shielding implies a greater increase in the magnetic energy: it is impossible without detailed calculations to be sure of the magnitude of the increase, but it is perhaps covered by a factor not exceeding 10.
7.8—An estimate of the order of magnitude of the velocity of $O$ can be made by considering the motion of a small element of the surface current-layer around $O$, supposing the resultant magnetic field near $O$ to be the same as if the stream-front remained plane. The actual field will be somewhat more intense than this, so that our estimated retardation will be somewhat too small.

The origin will now be taken at $C$ the centre of the Earth, and the axis of $z$ will be taken in the direction of motion of the stream. Thus

![Equatorial section](image_url)

the coordinate $z$ of $O$ will be negative while the stream is approaching the Earth, but $ds/dt$ will be positive.

Let $v$ denote the velocity of $O$ at time $t$, reckoned positively, so that $v=ds/dt$; also let $\rho_0$, $v_0$ be the density and velocity of the stream in the undisturbed regions behind the front, which are shielded from the magnetic field. Let $m$ denote the mass of the surface current-layer per unit area. Then, owing to the inflow of matter into the layer

$$dm/dt = \rho_0 (v_0 - v)$$  

The equation of momentum of the layer is

$$dmv/dt = F + \rho_0 v_0 (v_0 - v)$$

where $F$ denotes the electromagnetic retarding force per unit area, and the second term represents the flow of momentum into the layer from
behind. Supposing \( \rho_0 \) and \( v_0 \) to be constant, (31) and (32) may be combined to give
\[
\frac{dm(v - v_0)}{dt} = F
\]
The magnetic force in the layer varies from zero (or a low value) on the rear side to twice the normal value \( H \) of the Earth's field on the front of the layer; the mean value is therefore \( H \). The total current per unit surface-area of the layer, assuming complete shielding, is \( H/2\pi \), so that, apart from a factor of order unity,
\[
F = -\frac{H^2}{2\pi}
\]
Hence since at distance \( z \) from \( C \), \( H = \alpha^2 H_0/z^2 \), where \( H_0 \) is the equatorial value of \( H \) at the Earth's surface,
\[
\frac{dm(v - v_0)}{dt} = -\frac{H_0^2\alpha^6}{2\pi z^6}
\]
This equation is troublesome to integrate, and to avoid the difficulty we will introduce a factor \( v/v_0 \) on the right; this is equivalent to a reduction in the retarding force, and the results must afterwards be examined to see how far this departure from the actual conditions of the problem is likely to affect the results obtained.

Since \( v = dz/dt \), the solution of the modified equation
\[
\frac{dm(v - v_0)}{dt} = -\left(\frac{H_0^2\alpha^6}{2\pi v_0}\right)(1/z^6)(dz/dt)
\]
is
\[
m(v - v_0) = \frac{H_0^2\alpha^6}{2\pi v_0 z^6}
\]
no arbitrary constant being added since \( v = v_0 \) at \( z = -\infty \).

Integrating (31) we have also
\[
m = \rho_0 (v_0 t - \eta)
\]
In this equation we shall suppose that \((v_0 t - z)\) vanishes at \( z = -\infty \), so that \((v_0 t - z)\) represents the lag of the centre \( O \) of the stream-front, behind the position which it would have occupied had there been no magnetic field; at \( z = -\infty \) the magnetic field vanishes and there is no induced current or current-layer; for simplicity we take \( m = 0 \) at \( z = -\infty \), and no constant of integration is needed in (38). The vanishing of \((v_0 t - z)\) at \( z = -\infty \) implies that when \( O \) is there, \( t = -\infty \). The lag \((v_0 t - z)\) will be denoted by \( z' \), and both \( z \) and \( z' \) will be reckoned in Earth-radii \( a \), then being denoted by \( Z \) and \( Z' \). Thus
\[
Z = z/a, \quad Z' = z'/a = (v_0 t - z)/a = m/\rho_0 a
\]
and by (37) and (38)
\[
v/v_0 = 1 + 1/\gamma Z^4 Z'
\]
where
\[
\gamma = 10\pi \rho_0 v_0^2/H_0^2
\]
But
\[
dZ/dZ' = ds/ds' = v(dt/ds') = v/(v_0 - v) = -(1 + \gamma Z^4 Z')
\]
by (40).

By writing
\[
\zeta = \gamma Z^6/5, \quad \phi = dZ/dZ' = -(1 + \gamma Z^4 Z')
\]
so that
\[
v/v_0 = \phi/(1 + \phi)
\]
(42) is transformed to

\[ \frac{d\phi}{d\xi} = \frac{5}{6}[\phi(1+\phi)-\xi]/\xi \phi \]

If \( \phi \) can be determined from (45) as a function of \( \xi \), by (43) and (44) we know \( v \) as a function of \( \zeta \).

The graphs of the solutions of (45) for positive values of \( \xi \) consist of a series of curves as shown in Figure 6. The curves all touch the parabola \( \phi^2 + \phi - \xi = 0 \) or \( (\phi + 1/2)^2 = \xi + 1/4 \) at the origin. Some of them tend steadily to infinity; from this maximum they descend so as to cross the \( \xi \)-axis at right-angles, after which they converge again to the point \( \phi = -1 \) on the \( \phi \)-axis. These curves are wholly finite (for \( \xi > 0 \)); some of them have a minimum value lying on the same parabola.

The limiting solution which just goes to infinity is given by the asymptotic equation

\[ \phi = \sqrt{(5/2)\xi} - 5/7 + (\sqrt{10}/49)(1/\sqrt{\xi}) + \ldots \]

valid beyond about \( \xi = 3 \). The other solutions that go to infinity are those of the form \( \phi^2 = (5/2)\xi + A\xi^{3/2} \). At infinity the mass \( m \) of the conducting layer (per unit area) is small, and for analytical simplicity can for this purpose be treated as zero, so that there \( Z' = 0 \) (by 39); this condition is equivalent to \( A = 0 \), because

\[ Z' = -(1+\phi)/\gamma Z^b = -1/\gamma Z^b - (1/2\gamma Z^4 + A/5^{5/2}\gamma^{1/3})^{1/2} \]

and at \( Z = -\infty \) this is not zero unless \( A = 0 \).
The solutions near the origin are of the form
\[ \phi = \xi + (1/5)\xi^2 + \ldots \]
the first two terms in this power-series being common to all. The order of mutual contact of the curves at the origin is very high, the complementary function near the origin being approximately \( (B/\xi)e^{-\xi^2/16} \), which tends to zero very rapidly.

The graph of \( v \) as a function of \( z \) can be drawn for any value of \( \gamma \) by using the solutions (46) and (47) for large and small values of \( z \), and joining the two parts of the curve, the interpolation offering no difficulty. Owing to the vanishing of the constant \( A \), the nature of the solutions depends only on the parameter \( \beta \), which is proportional to the kinetic energy-density of the undisturbed stream.

In Figure 7 five curves are drawn to illustrate the dependence of \( v/v_0 \) upon \( \gamma \) (or \( z/a \)) for various values of \( \gamma \); the range over which the curves are drawn by interpolation is between \( v/v_0 = 0.1 \) and \( v/v_0 = 0.7 \). The values of \( \gamma \) are expressed in terms of the number \( N \) of hydrogen or calcium atoms per cc, moving with various velocities \( v_0 \), from \( 10^9 \) to \( 10^6 \) cm/sec, necessary to supply the corresponding value of \( \gamma \) or \( \rho v_0^2 \), taking \( H_0 = 0.3 \). The values of \( N \) vary inversely as \( v_0^2 \).

It must be remembered that the curves are only approximate, because the retarding force \( F \) is underestimated to the extent of the factor

<table>
<thead>
<tr>
<th>Curve</th>
<th>( N_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N_0 = 10^8 )</td>
<td>( \gamma = 10^8 )</td>
</tr>
<tr>
<td>I</td>
<td>0.01</td>
</tr>
<tr>
<td>II</td>
<td>0.10</td>
</tr>
<tr>
<td>III</td>
<td>1.00</td>
</tr>
<tr>
<td>IV</td>
<td>10.0</td>
</tr>
<tr>
<td>V</td>
<td>100</td>
</tr>
</tbody>
</table>
$v/v_0$. For $v/v_0 > 0.8$ the error is not serious, but for lower values of $v/v_0$ the curves must bend downwards increasingly more steeply than in the Figure. This, however, is not likely to make much difference in the distance at which $v/v_0$ is reduced to 0.1 (say), particularly in the case of the denser streams.

At first the decrease of velocity of the stream front is very slight, but from $v/v_0 = 0.9$ it is rapid until $v/v_0 = 0.1$, after which the further decrease to zero is gradual; this is because of the great mass then accumulated in the stream-front; it is at this part of the curve that the approximations made are most seriously in error, though not sufficiently to alter the general character of the result.

The following Table gives the calculated distance—an underestimate—at which $v/v_0$ has fallen to 0.1, in the case of hydrogen atoms for which $v_0 = 10^8$.

\[
\begin{array}{ccccccc}
N & Z = s/a & 0.01 & 0.1 & 1 & 10 & 10^5 \\
\hline
N_H & 0.01 & 10^{-4} & 0.025 & 2.10^4 \\
N_He & 0.10 & 10^{-3} & 0.25 & 2.10^3 \\
N_He & 0.01 & 2.50 & 0.025 \\
N_He & 1.00 & 1.0 & 2.50 & 0.25 \\
N_He & 100 & 1.0 & 250 & 2.50 \\
\end{array}
\]
It is clear that only in the case of the lower densities is the retardation sensitive to a change of density.

Figure 8 illustrates the time taken for the stream-front to approach the Earth. The time-origin has no special significance in this diagram, only the time-intervals being of importance. The straight line represents the travel of the stream-front if unaffected by the field; it corresponds to \( v_0 = 10^8 \text{ cm/sec} \), or about 10 Earth-radii per minute. The curved lines refer to the streams retarded as calculated; the retardation in time, at any distance, is measured by the difference between the straight line and the curve. The rapid bend in the curves corresponds to the sudden decrease in velocity mentioned above; after this stage the approach to the Earth is slow, especially for the denser streams, which, however, are but little retarded till they are within a few radii from the Earth.

7.9.—At \( Z \) Earth-radii, the Earth's magnetic intensity in the equatorial plane is \( 0.3/Z^2 \), while the change in the Earth's field due to an image-doublet at distance \( 2Za \) (corresponding to a distance \( Za \) for a plane conducting sheet producing complete shielding) is \( 0.3/(2Z)^3 \). This is of magnitude 30\( \gamma \) or 0.0003T when \( Z = 5 \); now 30\( \gamma \) is the order of magnitude of the initial rise of horizontal magnetic force during an ordinary magnetic storm; if we identify this rise with the field of the image-doublet here considered, it is of interest to consider how quickly this rise will occur. This is approximately the time taken for the stream-front to travel from \( Z = 10 \) (where at the Earth's centre the field of the image-doublet is \( 0.3/(20)^8 \) or about 4\( \gamma \)) to \( Z = 5 \). From Figure 8 it appears that this time is only about a minute, or less, if \( N > 10 \), but that if \( N < 1 \) it may be many minutes, or even hours. This calculation is faulty because the retardation has been underestimated, so that on this account the times mentioned must be somewhat lengthened (though by very little if \( N > 10 \)); on the other hand the increase in \( H \) due to the distortion of the front of the stream has not been allowed for. Further, the secondary effects of currents induced in the atmosphere of the Earth by the varying field of the image-doublet have not been considered. So far as the present calculations go, however, the value of \( N \) for storms having marked sudden commencements would appear to be greater than 10 for hydrogen atoms having velocities \( 10^8 \text{ cm/sec} \) (the corresponding limit for other types of ion and other velocities can be readily inferred).

(To be continued)