A MAGNETO-KINEMATIC MODEL OF THE SOLAR CYCLE

ROBERT B. LEIGHTON
California Institute of Technology, Pasadena
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ABSTRACT

A simple kinematical model of the solar cycle, based upon field amplification by the solar differential rotation, is presented. Numerical solutions of the model equations are found which are in close quantitative agreement with Spoerer's Law of Zones, Maunder's butterfly diagram, and the poleward migration of polar prominences. Fluctuations in eruption rate are introduced and are shown to produce fluctuations in period, amplitude, and relative phase and amplitude in the two solar hemispheres which agree closely with observed values.

I. INTRODUCTION

The nature of sunspots and of the 22-yr solar cycle has been a subject of speculation during the six decades since the discovery of magnetism in sunspots and of the polarity rules governing successive cycles (Hale 1908, 1913). An early idea held that sunspots are vortex tubes similar to terrestrial tornadoes, and that the magnetic field is somehow produced by this vortex motion. Today, we know that the magnetic field is the primary quantity, and that most of the properties normally associated with sunspots result from the presence of strong magnetic fields in the Sun's outer layers. Ideas about the cause of the solar magnetic cycle have also undergone considerable change. The periodicity has variously been attributed to the circulation of meridional mass currents in the Sun's outer convective layers, to the propagation of hydromagnetic waves through the Sun, or to torsional oscillations of the outer layers. A critical review of some of these ideas has been given by Cowling (1953). Of the numerous suggestions, Babcock's (1961) topological model, based on a mechanism discussed by Cowling (1953), seems to account in a natural way for some of the most characteristic qualitative features of the cycle, including Hale's polarity rules, the equatorward drift of average sunspot latitudes commonly referred to as Spoerer's Law of Zones (Carrington 1858; Spoerer 1894), and the now well-established alternation of sign of a large-scale "dipole" field. While much of Babcock's discussion was schematic and idealized, his model has undoubtedly advanced our understanding of the solar cycle.

Several authors have attempted to improve upon Babcock's model by using various forms for the isorotational surfaces (Chvojková 1965; Tuominen 1965; Vostrý 1967). Other workers have attempted to solve the hydromagnetic equations for a differentially rotating Sun (Nakagawa and Swarztrauber 1969). In common with Babcock's original discussion, these studies proceed from the assumption that, at some initial epoch, the subsurface magnetic field is purely dipolar; subsequent shear distortion of this poloidal field by the differential rotation then produces a toroidal field of increasing strength; at each latitude the field eventually becomes sufficiently strong (and thus buoyant) to erupt through the solar surface to form sunspots. Spoerer's law supposedly traces out the time of attainment of this critical eruption field strength as a function of latitude. The reversal of the magnetic field is supposed to result from the migration of large-scale magnetic regions from the sunspot latitudes to the polar regions. This field reversal is then assumed to leave the subsurface field in a pure poloidal form, ready to begin the next half-cycle.

Babcock's model assumes that, during the amplification stage, the time history of the
field at each latitude is governed solely by the differential rotation and is otherwise independent of the fields at other latitudes. However, the observed migration and expansion of magnetic regions, which carry the surface fields across zones of latitude, must modify the subsurface fields, and thereby must also modify the action of the differential rotation upon the fields. Since the migration proceeds on a time scale short compared with the period of the solar cycle, this source of cross-coupling of field components must be included in any quantitative discussion of the problem. Nakagawa and Swartztrauber (1968) included these effects through the introduction of a coefficient of magnetic diffusion, but were unable to follow the solution through more than a fraction of a cycle. Effects of field eruption were not included.

I have studied several examples of a semiempirical model of the solar cycle, based also on field amplification by the solar differential rotation. This model falls into a middle ground, between the essentially schematic form of Babcock's discussion and a rigorous solution of the hydromagnetic equations. The principal objective of this study is to invent a simple, quantitative, closed kinematical model of the solar cycle which is based as far as possible either on well-accepted physical effects or on observed facts (whether fully understood or not), in the hope that such a model might single out the most important factors for detailed study, and might suggest further directions for investigation. The results are sufficiently encouraging to justify preliminary discussion at this time.

The principal features of the solar cycle which the model describes are: (1) the existence of oscillatory modes for the subsurface fields, (2) the order of magnitude of the period, (3) the time variation of average sunspot latitudes (Sporer's law) and the width of the eruption zone (Maunder's butterfly diagram), (4) the time variation of the average radial magnetic field at all latitudes, (5) the irregular variations in the period and amplitude of successive cycles, and (6) the lack of perfect symmetry in the amplitude and phase of a cycle in the two solar hemispheres. The model also accounts qualitatively for the possible existence of persistent preferred longitudes of activity (Becker 1955) and the tendency of new spot groups to form within the boundaries of the \( \rho \)-parts of older magnetic regions (Tuominen 1962; Bumba and Howard 1964).

II. THE MODEL

Although the sporadic character of sunspot formation and the irregular magnetic patterns of individual spot groups show that significant fluctuations are present in the subsurface fields, the long-term regularity of the solar cycle suggests an underlying regularity in the field distribution which basically governs the cycle. For simplicity we therefore assume that the main features of the solar cycle are determined by the action of the solar differential rotation upon the three idealized, "average" field components \( B_r \), \( B_\theta \), and \( B_\phi \), which exist in a relatively thin "shear layer" of thickness \( H \ll \bar{R} \) in the outer part of the Sun \( \{ r, \theta, \text{and} \phi \} \) denote the radius, colatitude, and colongitude, respectively, in spherical polar coordinates). We disregarded the detailed variation of the magnetic and differential velocity fields throughout this layer, and assume that all relevant quantities either are constant or vary uniformly with depth through the layer.

The model may be outlined qualitatively as follows:

1. The fields studied are zonal and radial averages of the actual fields, and are assumed to vary only with latitude and time.
2. The differential rotation field is taken to be a given function of latitude and depth, purely azimuthal, and constant in time.
3. The differential rotation field acts upon \( B_r \) and \( B_\theta \) to change \( B_\phi \). Perfect conductivity and laminar flow are assumed.
4. If the total magnetic field \( B \) (practically, the component \( B_\phi \)) exceeds a certain critical strength \( B_c \), field eruption and the formation of sunspots are assumed to occur.
5. Eruption of the \( B_\phi \) field produces \( B_r \) fields having a net meridional magnetic moment (due to the systematic "tilt" of the axes of the spot groups).
6. The $B_r$ field is dispersed by a random walk (Leighton 1964); this dispersal is unaffected by the differential rotation for zonal average fields.

7. The eruption of the $B_\theta$ field reduces the strength of the remaining $B_\phi$ field.

8. Some minor, ad hoc factors which have a negligible effect on the steady state are introduced to hasten convergence of the model to a steady state and to improve the general behavior.

In the following paragraphs we discuss the rationale of the model and set out the basic equations which characterize it.

1. As discussed previously, zonal- and depth-averaged fields are used in order to simplify the problem. Of course, this precludes study of such things as preferred longitudes of eruption. Randomness of both latitude and time in the eruption process is included, however.

2. At the solar surface, the observed angular velocity of the differential rotation is approximately $\Omega_s = 18 \sin^2 \theta$ rad yr$^{-1}$ with respect to the polar regions (Newton and Nunn 1951). As a function of depth we assume

$$\Omega = \Omega_s + (a + \beta \sin^n \theta) \frac{R - r}{H},$$

where $a$, $\beta$, and $n$ are free parameters and where $R - H \leq r \leq R$.

3. On the assumption of perfect conductivity, the differential rotation field changes $B_\phi$ according to the equation

$$\frac{\partial B_\phi}{\partial t} = \sin \theta \left( B_\theta \frac{\partial \Omega}{\partial \theta} + R B_r \frac{\partial \Omega}{\partial r} \right).$$

With the assumed form for $\Omega$, we find

$$\frac{\partial B_\phi'}{\partial t} = \sin \theta \left[ -(a + \beta \sin^n \theta) \frac{R}{H} B_r' + \left(36 + n\beta \sin^{n-2} \theta \frac{R}{H} \right) \sin \theta \cos \theta B_\theta' \right].$$

In this equation, the primes signify that the field components are functions of $r$, $\theta$, and $t$. To average over $r$, we must make some assumption about the variation of $B_\phi'$ with depth through the layer. Two cases are considered: In case A, the shear layer is very thin compared with the depth to which the fields penetrate the Sun, and $B_\phi'$ is assumed zero in the shear layer; in Case B, the thickness of the shear layer is the same as the penetration depth, and $B_\phi'$ is assumed uniform with depth. In these two cases\(^1\)

$$\frac{\partial' B_\phi}{\partial t} = \sin \theta \left[-(a + \beta \sin^n \theta) \frac{R}{H} B_r \right] \text{ (case A)},$$

$$\frac{\partial' B_\phi}{\partial t} = \sin \theta \left[-(a + \beta \sin^n \theta) \frac{R}{H} B_r \right.$$

$$+ \left(36 + \frac{n\beta}{2} \sin^{n-2} \theta \right) \sin \theta \cos \theta B_\theta \right] \text{ (case B)}. $$

The prime signifies that the term comprises only one part of the actual time derivative.

4. The value of the critical field $B_c$ is regarded as an adjustable parameter. If $B_r$, $B_\theta$, $B_\phi$ correspond to the observed radial field at the surface; in case B, $B_r$ is half the surface field. Also, in case B, $B_\phi'$ would not be uniform with depth but would vary approximately linearly through the shear layer.
\[
\left( \frac{1}{2} g - 1 \right) \frac{\eta_0}{\epsilon} \frac{\partial \theta}{\partial t} = \frac{\eta_0}{\epsilon} \frac{\partial \theta}{\partial t}
\]

Thus, the characteristic time \( t^* \) is approximately \( t^* = \frac{\eta_0}{\epsilon} \).

We therefore, the equation for the time (Eq. 1960) as follows:

\[ t^* \approx \frac{\eta_0}{\epsilon} \theta^* \]

For the equation, we take the form:

\[ \frac{\partial \theta}{\partial t} - \frac{\partial^2 \theta}{\partial x^2} + \frac{1}{\eta_0} \frac{\partial \theta}{\partial x} = 0 \]

where \( \eta_0 \) is the diffusion time.

In the limit of \( \theta \rightarrow 0 \), the equation becomes:

\[
\left( \partial \theta - \theta \right) \partial^2 \theta_{\partial x^2} + \frac{1}{\eta_0} \partial \theta_{\partial x} = 0
\]

In the opposite sense, the total field \( \theta^* \), also combines to the field \( \eta_0 + \theta \), plus

\[
\frac{\partial \theta}{\partial t} - \frac{\partial^2 \theta}{\partial x^2} + \frac{1}{\eta_0} \frac{\partial \theta}{\partial x} = 0
\]

The equation of flux conservation is:

\[
\frac{\partial \theta}{\partial t} - \frac{\partial^2 \theta}{\partial x^2} + \frac{1}{\eta_0} \frac{\partial \theta}{\partial x} = 0
\]

Since the equation of conservation of flux is a field of \( \psi \), the field \( \psi \) is defined as:

\[
\psi = \frac{\eta_0}{\epsilon} \theta
\]

The quantity is considered constant for most of the process, and \( \frac{\partial \theta}{\partial t} \) is the rate of change of \( \theta \) with respect to time.

Concerning the quantification process, it is necessary to assume a value of \( \eta_0 \) at which the field is defined. Hence, a field of electromagnetic fields is defined, and \( \psi \) is used to correspond to the field of \( \psi \).

\[ \psi = \frac{\eta_0}{\epsilon} \theta
\]

When the field is defined, the field \( \psi \) is defined, which is the only observable field component, is scaled to agree appropriately with the observations.
However, it is necessary to allow for the fact that large spot groups (that is, large fluxes) correspond to greater values of $a$. If we take $a_0$ as the linear extent of a spot group corresponding to $|B_\phi| = B_s$, it is convenient to assume that $a = a_0 |B_\phi|/B_s$. Then we have

$$\frac{\partial'' B_\phi}{\partial t} = -a_0 |B_\phi| B_s / 2 \pi R B_s \tau,$$

(6)

where, numerically, we assume that $a_0$ is about one-third the extent of an "average" spot group, so that $a_0/2\pi R \approx 1/100$. A corresponding allowance for the variation of $a$ need not be made in evaluating the meridional magnetic doublet moment, since the tilt angle $\gamma$ is smaller for larger groups, tending to cancel the effects of changing $a$. We assume that $a \sin \gamma = \text{const.}$ at a given latitude.

8. Equations (3)–(6), when combined, are the basic equations of the model ($B_\phi$ is deduced from $B_\tau$ via the equation $\nabla \cdot B = 0$). In practice, however, certain problems arose which led to the introduction of further factors:

a) At a given latitude, $|B_\phi|$ can be less than $B_s$ in one half-cycle, but can become greater than $B_s$ in the other half-cycle, rendering the cycle asymmetric in its time behavior. This asymmetry usually disappears slowly with time, but may persist for many cycles. To reduce such effects, a slow exponential decay of $B_\phi$ was introduced. The time constant was chosen to be 50 years—long compared with $22/2\pi$, but short compared with the 175-yr average interval covered in the execution of the computer program.

b) A second effect of the assumption of a threshold field $B_s$ was that, if the amplitude of oscillation became less than a certain value, the oscillation would die out altogether, because of the random-walk term. While it is possible that the real Sun might behave in this way, the possibility seems remote in view of the several occasions on which one hemisphere or the other has remained nearly free of spots for many years. According to Wolf and Sporer (Maunder 1922), both hemispheres remained almost bare of spots for nearly seven decades near 1700.

To avoid this behavior, a small fraction $G$ of the radial field (usually about 0.003) is assumed to escape dispersion by the random walk, and to be produced at a rate proportional to $B_\phi$. This "unruptured" radial field, called $B_s$, is also assumed to decay with a 50-yr time constant so as to hasten the approach of the oscillation to a symmetrical steady state.

c) Because of the solenoidal character of $\mathbf{B}$, the radial fields $B_\tau$ and $B_\phi$ should give zero net flux through the solar surface. However, because of the finite spatial and temporal steps used in solving the differential equations, a small but finite net flux may appear. To avoid this effect, the net flux was evaluated after each iteration and the average (constant) field was subtracted at all latitudes.

d) A factor $F$ of order unity was introduced into equation (4) to allow for the possibility that $a$ or $\gamma$ might be incorrectly evaluated from the available data. For example, the assumption of an 8° longitude span for an "average" spot group might be an overestimate because of the very short lifetime (less than 1 day) of most spots. Such a factor was found necessary in order to obtain oscillatory solutions for some models. In most cases, $F$ was given the smallest value that would still insure an oscillatory solution.

To summarize, the solar-cycle model is defined by the following equations, in which $B_\tau$, $B_s$, $B_\theta$, and $B_\phi$ are functions of $\mu = \cos \theta$ and $t$, $t$ being measured in years:

$$B_\theta = \frac{R}{H \sin \theta} \int_{-1}^{\mu} (B_\tau + B_s) d\mu,$$

(7)

$$\frac{\partial B_\phi}{\partial t} = \sin \theta \left[ -(\alpha + \beta \sin^\alpha \theta) \frac{R}{H} (B_\tau + B_s) + \epsilon \left( 36 + \frac{n^2}{2} \sin^{n-2} \theta \right) \right. \times \sin \theta \cos \theta B_\theta - \frac{B_\phi}{100 B_s \tau} - \frac{B_\phi}{50},$$

(8)
\[
\frac{\partial B_z}{\partial t} = -\delta \frac{FH}{80R^2} \frac{\partial}{\partial \mu} (\mu B_\phi) + \frac{1}{T_D} \frac{\partial}{\partial \mu} \left[ (1 - \mu^2) \frac{\partial B_z}{\partial \mu} \right],
\]
(9)

\[
\frac{\partial B_z}{\partial t} = -\frac{GH}{80R^2} \frac{\partial}{\partial \mu} (\mu B_\phi) - \frac{B_z}{50}.
\]
(10)

In these equations \( \epsilon = 0 \) for case A and \( \epsilon = 1 \) for case B; \( \delta = 0 \) if \( |B_\phi| \leq B_0 \), and \( \delta = 1 \) otherwise; \( a/4\pi R = 1/80 \), \( a_0/2\pi R = 1/100 \), and \( \sin \gamma = \mu/2 \) have been substituted. In equations (7) and (8), \( \sin \theta = \sqrt{(1 - \mu^2)} \) and \( \cos \theta = \mu \).

III. PRELIMINARY DISCUSSION

Although the equations appear to contain nine adjustable parameters, not all of these affect the qualitative character of the model.

The decay time \( T_D \) for the random walk is known within about 50 per cent from the scale and lifetime of the supergranulation. It is set equal to 20 yr for most of the present calculations (Leighton 1964), although the effects of its variation are studied.

The thickness \( H \) of the shear layer is unimportant—it merely sets the scale of \( B_\phi \) and \( B_z \) relative to that of \( B_r \) and \( B_\psi \). (The value of \( H \) is of importance, however, in connecting measured field strengths of sunspots with measured fluxes of spot groups by using the model.)

The critical field \( B_z \) affects only the amplitude of the oscillation. For each case \( B_z \) was assigned a value which yielded a convenient numerical amplitude.

As discussed previously, the coefficient \( G \) is supposed to be much smaller than unity, but sufficiently large to cause spontaneous oscillation. The value \( G = 0.003F \) was found suitable and was adopted for all examples studied. (If \( G \) is taken as large as 0.1\( F \), the non-dispersed field \( B_z \) often dominates the situation, and the solution loses its physical interpretability.)

Thus the model is practically defined by the five parameters \( \alpha, \beta, \gamma, \tau, \) and \( F \). For the case \( \alpha = \beta = 0 \), only the latitude dependence of \( \Omega \) plays a role, and there are effectively only two parameters, \( \tau \) and \( F \).

The procedure used in solving equations (7)–(10) for given \( \alpha, \beta, \gamma, \) and \( \epsilon \) was to begin at \( t = 0 \) with an arbitrary \( B_\psi(\mu) \) field and its corresponding \( B_\phi(\mu) \) field, but with no azimuthal field \( B_\phi(\mu) \) or radial field \( B_r(\mu) \), and to follow the time development of the fields by numerical integration until a steady state was reached. The first objective was to obtain a stable oscillatory solution having a period of 22 yr by adjusting \( F \) and \( \tau \).

From equation (9) it is apparent that \( F \) and \( \tau \) will have opposite effects on the period. Physically, it is clear that \( F \) must also strongly govern the general character of the solution: If the solution is to be oscillatory, the erupted flux must produce sufficient axial dipole moment not only to cancel the previously existing moment but also to establish an equal one of opposite sign, all in the presence of the dispersive effects of the random walk. Thus, as one would expect, oscillatory solutions result only for values of \( F \) greater than a certain minimum value \( F_m \) for each given case. This minimum value is of some interest as an indicator of the general acceptability of a particular version of the model, for if \( F_m \) greatly exceeds unity, it would seem that much larger axial tilts or major-axis dimensions of the spot groups would be needed than are assumed. Parameter choices which give smaller values of \( F_m \) would be favored.

IV. NUMERICAL CALCULATIONS

In order to hold the computation time to a minimum, almost the largest temporal and spatial steps compatible with interpretability and stability of the numerical solution were used; typical values were \( \Delta t = \frac{1}{12} \) yr and \( \Delta \mu = 0.1 \).
To find $\tau$ and $F_m$, the computation was started by using values of $F$ and $\tau$ which were fairly sure to give an oscillatory solution. The period was automatically evaluated, and after each cycle a new value of $\tau$ was used which would move the period toward 22 yr. To find $F_m$, the values of $F$ and $\tau$ were slowly reduced as the computation progressed, the value of $\tau$ being corrected after each full cycle, as described. At some value of $F$ the oscillation would die out. This value was used as an approximation to $F_m$. Then, with $F$ fixed at some convenient value somewhat greater than $F_m$, a final value of $\tau$ was determined and the steady-state solution was studied. For convenience, a numerical printout format resembling a contour plot was used. Each of the 101 characters of a printed line represented a rounded value of a field component on a scale of 0–9, positive fields being represented by numerals and negative fields by alphabetical letters A–I. Fields greater than +9 or less than –9 were represented by + or –, respectively. In rounding, if a given value fell more than ±0.25 unit from an integer, a blank character was printed. Values at points lying between the twenty-one values of $\mu$ used in the calculation were found by linear interpolation.

V. RESULTS

a) Cases Having No Radial Angular Velocity Gradient

The first case to be studied was the case $a = \beta = 0$, $\epsilon = 1$. The minimum value of $F$ for which previously existing oscillations could be maintained was approximately $F_m = 6$. A value of $F = 10$ was used in the calculations. The corresponding value of $\tau$ was $\tau = 0.42$ yr. The quantity $B_\phi$ was set equal to $20 R/H$. Typical printouts of $B_\theta$, $B_\phi$, and $B_r$ are shown in Figure 1, and corresponding (smoothed) contour diagrams of $B_\phi$ and $B_r$ in Figure 2.

This case exhibits several features characteristic of nearly all versions of the model so far studied. The most striking feature is the appearance of distinct “butterfly diagram” forms for the field components $B_r$ and $B_\phi$. Because of the role of $B_\phi$ in defining the rate of field eruption, the $B_\phi$ diagram may be regarded as equivalent to Maunder’s original plot of sunspot occurrence, and, although a quantitative correspondence is not to be expected on several grounds, the $B_\phi$ map appears to provide a reasonably good description of the time variation of the latitude zone of sunspot activity. The time variation of the zone of maximum activity is compared with Spoerer’s law in Figure 3.

Agreement with the usual form of the butterfly diagram is improved if one takes the contour corresponding to about 50 per cent of the maximum of $B_\phi$ as defining the limits of actual sunspot eruption. Such a contour is emphasized in Figure 2. This choice, incidentally, also agrees well with observed time intervals from minimum to maximum and return to minimum. These times are, respectively, 4.6 and 6.4 yr for this version of the model, as compared with 4.50 and 6.56 yr obtained by averaging over eighteen actual solar cycles (Waldmeier 1941).

A second characteristic feature shown by this case is the high-latitude poleward trend of the zero-field line of the $B_r$ plot. According to Waldmeier (1960), the zero-field line marks the path of the cyclic poleward migration of the polar prominences. From the plot of Ananthakrishnan (1954) (see de Jager 1959), one finds that the polar prominence zone moves poleward at a rate $d\mu/dt = 0.085$ yr$^{-1}$ and arrives at the pole 3.6 yr after solar minimum, under average conditions. Although this effect is poorly defined in this example, the poleward rate is approximately correct, and the arrival time is about 3 yr after minimum.

Serrated shapes are exhibited by some of the contours in nearly all versions of the model. These are most pronounced where the contours are rather widely spaced but lie near the boundary of a steep gradient of the field. This effect is an artifact resulting from the rather coarse intervals used in the calculation. To reduce it, the final calculations for each case were made using intervals $\Delta \mu = 0.05$ and $\Delta t = 1/48$ yr.
\[ B_\phi = \alpha = \beta = 0 \]

Fig. 1.—Computer solution of solar-cycle model for the case of no radial angular velocity gradient. (a) \( B_\phi \); (b) \( B_r \), and (c) \( B_\theta \). Field strength per unit interval is \( 8R/H \) units for \( B_\phi \), 2 units for \( B_r \), and \( 2H/2H \) units for \( B_\theta \). Critical field \( B_c \) is \( 20R/H \). The solution used, \( \Delta t = 0.1 \), \( \Delta t = 1/12 \) yr, \(-1 \leq \mu \leq 1\), is plotted horizontally, and time increases at the rate of \( \frac{1}{y} \) yr per line from top to bottom.

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\[ \alpha = \beta = 0 \]
Fig. 2.—Contour plots of (a) $B_\phi$, (b) $B_r$, for the case of no radial angular velocity gradient. The solution for this case was calculated by using $\Delta \mu = 0.05$ and $\Delta t = 1/48$ yr. Contour intervals are the same as for Fig. 1. Dashed line defines the limits of "zero" field on the numerical printout, and illustrates the degree of smoothing used in drawing the contours.
b) Cases Having a Radial Angular Velocity Gradient

In spite of its simplicity, the above basic case is in remarkable agreement with many of the observed features of the solar cycle. Nevertheless, it is not completely satisfactory because of the rather large value of $F$ which must be used to guarantee oscillation. For this reason as well as on general grounds, we are led to consider other cases involving radial variation of rotation rate.

Consideration of the relative signs of $B_r$ and $B_\theta$ for a dipole field, and inspection of equation (3b), show that $\alpha > 0$ and/or $\beta > 0$ will tend to increase the rate of production of $B_\theta$ for a given $B_r$ and $B_\phi$. Since greater production of $B_\theta$ will lead to greater eruption rates and therefore greater meridional magnetic moments, we may expect that positive values of $\alpha$ or $\beta$ will decrease the size of $F_m$. Positive $\alpha$ or $\beta$ correspond to an inward gradient of angular velocity—that is, to the interior rotating faster than the exterior.\footnote{If the interior were to rotate more slowly than the exterior, the oscillatory solutions would have the “butterfly” plot starting at low latitudes and proceeding toward the poles.}

![Graph showing the relationship between $\cos \theta$ and years for various model cases.](image)

**Fig. 3.—Comparison of computer solutions for solar-cycle model with Spoerer’s Law of Zones for several cases.** Solid curve with error bars represents Spoerer’s law and its uncertainty for an average sunspot cycle according to Waldmeier (1939). The various symbols denote the latitude of maximum $B_\phi$ for various model cases.

These expectations are well borne out by actual calculations. In fact, the solution is quite sensitive to the radial gradient, so much so that it seems likely that the actual solar cycle is governed principally by the interaction of a radial gradient of angular velocity with the radial field.

As the next case for study, the values $\alpha = 0$, $\beta = 18$ rad yr$^{-1}$ were chosen. (This corresponds to a variation in angular velocity through the shear layer at the equator equal to the variation at the surface from pole to equator. It also corresponds to Keplerian motion of the material in an equatorial layer whose thickness $H$ is approximately one-tenth the solar radius.) Cases A and B were studied for various values of $n$ from 2 to 10. All cases exhibit features similar to those described above for the basic case $\alpha = \beta = 0$, but differ in detail. The principal features are compared in Figure 4 and Table 1. In this table, $F_m$ is the minimum value of $F$ for which oscillations could be sustained; $F$ is the value used in the calculation; $\tau$ is the value which gives a 22-yr period for this value of $F$; $B_{\text{rpm}}$ is the maximum value of $B_r$ attained at the pole, set equal to 1 gauss;
### Table 1

| ℓ | n | a | b | c | d | e | f | g | h | i | j | k | l | m | n | o | p | q | r | s | t | u | v | w | x | y | z |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| 3.6 | 0.01 | 0.05 | 0.10 | 0.15 | 0.20 | 0.25 | 0.30 | 0.35 | 0.40 | 0.45 | 0.50 | 0.55 | 0.60 | 0.65 | 0.70 | 0.75 | 0.80 | 0.85 | 0.90 | 0.95 | 1.00 | 1.05 | 1.10 | 1.15 | 1.20 | 1.25 | 1.30 | 1.35 | 1.40 | 1.45 |

*Note: The table represents a comparison of various models.*

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$B_{rm}$ is the maximum value of $B_r$ attained during the cycle; $B_{\phi m}$ is the maximum value of $B_\phi$ attained during the cycle; $B_\phi$ is the critical field strength; $\mu_s$, $\mu_m$, and $\mu_c$ are the values of $\cos \theta$ corresponding to the start, the middle (maximum), and the end of the sunspot cycle (the boundary of sunspot eruption is here defined by the $B_\phi$ contour corresponding to half the maximum value); $\mu_u$ and $\mu_l$ mark the upper and lower limits of sunspot activity; $T_F$ is the time at which the radial field changes sign at the pole. $T_{Erm} - T_M^*$ is the difference in time between the epochs of maximum $B_r$ and maximum $B_\phi$.

![Graph](image.png)

Fig. 4.—Comparison of computer solutions for the case $a = 0, \beta = 18$ rad yr$^{-1}$ for various values of $n$. Closed contours mark the locus of the $B_\phi$ field at half the maximum value. Tilted solid lines mark the locus of zero radial field. Dashed line with error bars defines the observed rate of poleward migration of the polar prominences, as deduced from the plot of Ananthakrishnan (1954). The corresponding contours for the case of no radial angular velocity gradient ($a = \beta = 0$) are also shown.

The results show clearly that in terms of the value of $F$ needed to maintain oscillation, a radial gradient of angular velocity is 10 times as effective, for a given $\Delta \Omega$, as a latitude gradient. Values of $F$ near 0.6 suffice to maintain oscillation for the cases involving 18 rad yr$^{-1}$ radial variation of $\Omega$, as compared with the value 6 found necessary for the basic case $a = \beta = 0$. Thus the model is able to provide oscillatory solutions if parameter values are used which correspond with the observed properties of sunspots, but only on the assumption that the interior of the Sun rotates several radians per year faster than the outer layers.

As $n$ is increased from 2 to 10, the solutions show certain systematic trends. Since a higher value of $n$ has the effect of confining the differential rotation toward the equator, the butterfly diagram is thereby compressed toward lower latitudes. The epoch of polar arrival of the zero-line of radial field is forced to a later time for larger $n$, but the rate at which the zero-line approaches the pole is little affected by $n$, since this rate is governed mostly by the random-walk process (Leighton 1964).

Case A solutions ($\epsilon = 0$) were generally similar to case B solutions ($\epsilon = 1$), except that the former generally required somewhat greater values of $F$ for oscillation, especially
for the larger values of \( n \). This is qualitatively reasonable because the \( B_r \) and \( B_\theta \) terms of equation (8) reinforce each other.

c) Rigid-Body Core

In the cases so far studied, a latitude variation of differential rotation has been present throughout the shear layer. However, there is considerable interest in models in which the bottom of the shear layer is in rigid rotation (Haurwitz 1968; Dicke 1964). Such cases correspond to parameter values of \( \beta = -18, n = 2 \), and any value of \( a \). Two such cases are of special interest: \( a = 18 \) and \( a \) very large, say \( a = 1000 \). The former value gives a period of rigid-body rotation equal to that at the surface at the equator, about 27\(^{d}\) (synodic), while the latter corresponds to a rapidly rotating core.

Oscillatory fields were found for both of these cases. However, the butterfly diagrams and numerical parameter values needed to produce the necessary 22-yr period were not very satisfactory. In both cases the zone of sunspot eruption occurred at much too high latitudes. In the case \( a = 18 \), eruption began at \( \pm 65^\circ \), and at sunspot maximum the active zone was centered at \( \pm 45^\circ \); in the case \( a = 1000 \), eruption started at \( \pm 55^\circ \), and at sunspot maximum the active zone was centered at \( \pm 30^\circ \). Furthermore, in the latter case, the value of \( \tau \) needed to attain a 22-yr period (for \( F = 1 \)) was \( \tau = 83 \) yr, and the maximum \( |B_\phi| \) was 2000 times as great as the critical field \( B_c \). One cannot interpret these values physically on the basis of the present model, for, if \( \tau \) greatly exceeds the period of the solar cycle, one would assume that several “sheets” of oppositely directed field were present at the same time, each slowly rising to the surface. However, such an idea is difficult to reconcile with the averaging over depth on which the model is based. Again, the extremely large values of \( |B_\phi|/B_c \) obtained are difficult to reconcile with observed sunspot fields, which range in strength by only a factor of 10, from perhaps 500 to 5000 gauss. (In many of the previously studied cases, one can rather consistently relate \( B_c \) to the minimum sunspot field and \( |B_\phi|_{\text{max}} \) to the maximum field, as will be discussed later.)

d) Effects of Other Assumed Forms for Equations

Next, the character of the solution was studied as the form of one of the equations was changed, to find out whether the quasi-linear character of the model plays an essential role in the kind of solution obtained. The \( \delta \)-terms appearing in equations (8) and (9) were modified by replacing \( \tau \) by \( \tau_0(B_c/|B_\phi|)^m \). Setting \( m > 0 \) makes the eruption time constant shorter for stronger fields. Oscillatory solutions were found for \( m = \pm 1 \) and \( m = 1 \); these were generally similar to those for \( m = 0 \) except for the numerical values of \( F \), \( \tau \), and \( B_c \) required. In the case \( m = 1 \), rather large values of \( F \) and \( \tau \) were needed to attain oscillation at the desired 22-yr period. The effect of removing the condition that \( |B_\phi| \geq B_c \) for eruption was also tested. With \( B_c \) left the same, the equations were solved without interposing this condition. As was expected, oscillation occurred with a shorter period \( (T \sim 9 \text{ yr}) \) and starting at higher latitudes \( (\mu \sim 0.7) \). When the period was made longer by increasing \( \tau \), the solution became damped at about a 16-yr period. Several attempts were made to produce a stable oscillation with a 22-yr period by substantially increasing \( F \), but without success. The reason for this difficulty was not determined.

The dependence of \( a \) upon \( |B_\phi| \) (see eq. [6]) was modified to the more general form \( a = a_0(|B_\phi|/B_c)^p \) with \( p = \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \), or (in the “standard” case) 1. These changes produced no change in the general character of the solution, and very little change in the numerical values obtained.

e) Symmetric Modes

In the study of the case \( a = 0, \beta = 18, n = 2, \epsilon = 0 \), an interesting situation arose in which the \( B_\phi \) solution, which had started as an antisymmetric function of \( \mu \), slowly
changed into a symmetric function of $\mu$ with a comparable period. The configuration of the radial field is dipole-like in the first case and quadrupole-like in the second. The possible existence of a symmetric mode was investigated for a few other cases, using the same parameter values as those which gave a 22-yr period for the antisymmetric oscillation. In most instances a symmetric oscillation having a period comparable to 22 yr was observed, but this oscillation usually decayed away or reverted into an antisymmetric oscillation over an interval of a few cycles. The significance of this symmetric mode, even though it is not usually the dominant one, is that fluctuations in the eruption of flux in the two solar hemispheres will often lead to a small amount of amount of symmetric field, which might then persist for a few sunspot cycles as the dominant antisymmetric and the decaying symmetric modes "beat" with one another (Newton and Milson 1955; Bell 1962).

Typical "butterfly" diagrams for the $B\phi$ and $B_r$ fields of the symmetric mode $\alpha = 0$, $\beta = 18$, $n = 2$, $\epsilon = 0$ are shown in Figure 5.

f) Adoption of a "Standard" Case

Many of the solutions discussed so far are in good qualitative agreement with known features of the solar cycle, and some are in rather good quantitative agreement. While close agreement with observation cannot be taken as establishing the detailed validity of the present rudimentary model or as establishing the numerical magnitudes of its parameters, those conditions which give closest agreement might suggest certain factors to be of importance for further study. A quantitatively accurate model is also of value as a starting point for the inclusion of previously neglected effects. In this spirit, the cases $\alpha = 0$, $4 \leq \beta \leq 36$, $6 \leq n \leq 8$, $\epsilon = 1$ are found to agree reasonably well with observation. The lower values of $\beta$ require $F$-factors significantly greater than unity, which seems undesirable. Larger values of $\beta$ permit smaller $F$-factors but require somewhat greater values of $\tau$. Good agreement with Spoerer's law is found for $\beta \approx 10$ (see Fig. 3). Smaller values of $n$ give eruption at too high latitudes; larger values, at too low latitudes. Cases with $\epsilon = 0$ require relatively large values of the $F$-factor, for the $n$-values of interest. Thus the case $\alpha = 0$, $\beta = 10$, $n = 8$, $\epsilon = 1$ is chosen as a reasonable one for further study. Contour plots of $B\phi$ and $B_r$ for this case are shown in Figure 6.

g) Introduction of Randomness

The above "standard" model was next extended to include a random factor in the eruption rate, to see whether the resulting solutions would show some of the variability which characterizes the actual solar cycle. The random factor was introduced in the form of a variable eruption time "constant" $\tau$. The value of $\tau$ at each latitude was chosen at random three times each year from a table of twenty-one equally likely values distributed according to an assumed Gaussian distribution of $\ln \tau$. The average $\ln \tau_0$ and variance $\sigma^2$ of the distribution were chosen to correspond to an average eruption period of 11 yr and an rms fluctuation in eruption period of 1-2 yr, to agree with the available data on the sunspot cycle (Waldmeier 1941). The parameter values needed to achieve this were: $F = 2$, $\tau_0 = 0.60$ yr, $\sigma = 1.0$. The $\tau$-values used thus ranged from 0.133 to 2.12 yr.

With these parameter values, a few examples of twenty successive 11-yr cycles were computed, and their statistical properties were compared with similar data for eighteen actual solar cycles. Although precise comparison is impossible owing to the empirical nature of the conventional measures of solar activity (Wolf numbers, sunspot areas, etc.), the agreement between the erupted magnetic flux according to the model and the tabulated activity indices for the Sun is remarkable. Typical butterfly diagrams for the model are shown in Figure 7, $a$, and may be compared with contour plots of sunspot areas (Fig. 7, b) taken from Becker (1955). A comparison of various quantities derived
Fig. 5.—Contour plot showing symmetric or quadrupole mode for the case $a = 0, \beta = 18, n = 2, \epsilon = 0$. (a) $B_\phi H/R$; (b) $B_r$. The parameters $F$ and $\tau$ were fixed at the same values which yielded a 22-yr period for the antisymmetric or dipole solution.
Fig. 6.—Contour plots of $B_{\phi}$ and $B_r$ for the "standard" case $\alpha = 0$, $\beta = 10$, $n = 8$, $\epsilon = 1$. (a) $B_{\phi}H/R$; (b) $B_r$. 
Fig. 7.—Butterfly diagram of eruption rate for model with fluctuations. The format of this plot is as follows: Cols. 1 and 2: Time in years (every third line); cols. 4–105: eruption rate as a function of cos θ, for $-1 \leq \cos \theta \leq +1$ (see text); cols. 107–109: 4-month average flux erupted S. of equator; cols. 110–112: 4-month average flux erupted N. of equator; cols. 114–117: 1-yr average erupted flux; cols. 120–122: 1000 cos θ, where θ is the average colatitude of the erupted flux; cols. 125–127: 1-yr average erupted flux S. of equator; cols. 130–132: 1-yr average erupted flux N. of equator. In the columns, the numbers in the basis of the random variables were selected. The erupted fluxes must be multiplied by $\kappa R^3 \Delta \theta$, where $\Delta \tau = \frac{1}{2}$ yr, $\Delta \theta = 0.1$, and $\nu$ is a factor, approximately equal to 1.5, needed to normalize to $B_{\text{rpm}} = 1$ gauss.

18
from the model and from actual solar data is made in Table 2. The first quantity, the average interval between successive minima and its fluctuation, serves to fix the parameters $t_a$ and $\sigma$. The remaining quantities are then determined. All but the last are in excellent agreement with actual solar values. The last quantity, the fluctuation in the ratio of strengths at sunspot maximum in the northern and southern hemispheres, is somewhat less for the model than for the Sun, and it typifies the fact that the short-term fluctuations produced by the model are usually less than those shown by the Sun. This may be caused by the fact that erupted fluxes in the model are compared with sunspot areas or numbers on the Sun. The latter, being but a transient manifestation of the much more stable and long-lived magnetic flux of an active area, are probably subject to wider variations. The variation of eruption rate according to the model, as compared with yearly averaged sunspot area, is shown in Figure 8.
A correlation between period and amplitude was sought by comparing the average periods for cycles whose amplitudes were greater than, or less than, average. Only a small effect of marginal significance was found for either the model or the Sun, but this small effect was in the same sense in both cases, as shown in Table 3. In the table, $T_{GA}$ is the period for cycles whose amplitude was greater than average, and $T_{LA}$ is that for cycles of smaller than average amplitude. The stated uncertainties are estimated standard deviations of the mean.

Among the well-established solar-cycle correlations are the dependence of cycle development on maximum amplitude (Waldmeier 1957a) and the dependence of the latitude motion of the sunspot zone on the maximum amplitude of a cycle (Waldmeier 1939). These correlations were evaluated for the model in a manner similar to that used by Waldmeier for the solar data, and are shown in Figures 9 and 10. The curves of Figure 9 differ from Waldmeier’s result that high-maximum solar cycles rise faster to maximum than do low-maximum cycles. Also, the higher curves in the model fall faster after maximum than do the lower ones, rather than decaying at about equal rates as is observed. Similarly, while the curves of Figure 10 show that strong cycles occur at higher average latitudes than do weak ones, the detailed shapes of the curves differ from the simple parallel displacement found by Waldmeier.

Other statistical properties of interest include correlations from cycle to cycle of the strength of activity and of the dominance of one hemisphere over the other. Numerous investigators have attempted to analyze the rather limited solar-activity data for persistent effects or for long-time periodic behavior. The possibility of an 80-yr periodic modulation of the 11-yr cycle of solar activity has been much discussed. On the basis of twenty cycles of data one cannot clearly demonstrate either randomness or order, either

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**Fig. 8.**—Comparison between yearly average eruption rate of magnetic flux according to the model, and observed yearly average sunspot areas.

**TABLE 3**

<table>
<thead>
<tr>
<th></th>
<th>$T_{GA}$ (yr)</th>
<th>$T_{LA}$ (yr)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sun......</td>
<td>10.8 ± 0.5</td>
<td>11.3 ± 0.5</td>
</tr>
<tr>
<td>Model......</td>
<td>10.5 ± 0.3</td>
<td>11.6 ± 0.3</td>
</tr>
</tbody>
</table>

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Fig. 9.—The development of a cycle as a function of average maximum amplitude. I, II, III refer respectively to the lowest, middle, and highest thirds of the cycles, in terms of the maximum amplitude attained.

Fig. 10.—Dependence of the latitude variation of the sunspot zone on the maximum amplitude of a cycle. I, II, III refer to the same groups of cycles as in the previous figure.
for the Sun or for the model, using elementary tests for statistical correlation of successive cycles. The question presumably could be answered for the model simply by increasing sufficiently the number of cycles of computation. Superficially, both the Sun and the model seem to show a smaller fluctuation of amplitude from cycle to cycle than would be expected if each cycle were independent. The correlation coefficient of maximum amplitude for successive cycles is $0.23 \pm 0.2$ for the Sun and $0.5 \pm 0.2$ for the model. Similarly, the ratio of activities in the two hemispheres seems to be persistent. For the Sun, the correlation coefficient of the ratio of northern to southern activity for successive cycles is $0.57 \pm 0.3$, while that for the model is about $-0.1 \pm 0.1$. Clearly, these properties of both the Sun and the model deserve more careful study.

$\textit{h) Variation of } T_D$

The characteristic time $T_D$ was set equal to 30 yr to find the effect of this change on the solution. The only significant change was in the rate at which the zero-field line of $B_\phi$ approached the pole and the epoch of arrival at the pole: The approach was approximately two-thirds as rapid, and the arrival was correspondingly later.

$\textit{i) Variation of Initial Conditions}$

Most of the solutions were started by assuming $B_\phi(\mu) = \mu \sqrt{(1 - \mu^2)}$. However, it was verified that oscillation could be reliably initiated using fields less than 10 per cent as strong. Also, a satisfactory oscillation was initiated by assuming $B_\phi(\mu) = 0$ except at $\mu = +0.5$, where $B_\phi(0.5) = 1$.

$\text{VI. SUMMARY}$

In the preceding section we have shown that a relatively simple model provides a surprisingly accurate and broad description of the solar cycle. Oscillatory solutions are found for a wide variety of assumed forms for the differential rotation. The best agreement with observation is found if one assumes a moderate amount of radial variation of rotation rate, with the interior of the Sun rotating more rapidly than the surface, and with rather strong concentration of this radial shear toward the equator.$^3$ The value of the eruption time constant $\tau$ which yields the proper period is on the order of a few months.

Field maps of $B_\phi$ very similar to Maunder’s butterfly diagram are characteristic of all antisymmetric periodic solutions so far studied. The best agreement is found if it is assumed that fields weaker than about 40 per cent of the maximum value do not actually appear as sunspots but otherwise behave in the same way as fields that do form sunspots. The implications of this idea regarding the subsurface field geometry are not yet clear.

The radial field $B_r$ also appears in a form resembling Maunder’s butterfly diagram. The $B_r$ butterfly typically approaches the equator more rapidly and reaches its maximum 1–3 yr before the $B_\phi$ field. These properties can eventually be checked observationally by using synoptic magnetic data. The poleward motion of the zone of zero radial field matches qualitatively the observed migration of the zone of polar prominences.

In addition to the antisymmetric or dipole-like mode, a (usually damped) solution is also commonly found having a symmetric or quadrupole-like character and a period of the same order as but usually somewhat longer than that for the dipole case. This raises the interesting possibility of simultaneous excitation of both kinds of modes by the random character of sunspot eruption.

Random eruption rates are easily included in the model and lead to behavior which matches many of the fluctuations characteristic of the solar cycle.

$^3$ If the rotation becomes uniform at some depth within the Sun, we assume that this occurs at a level below that within which the magnetic field is concentrated.
VII. DISCUSSION

a) General Properties of the Solutions

Certain general aspects of the solution can be identified with various terms appearing in the model equations. Thus the equatorward “drift” of the sunspot zone results directly from the mutual effects of the δ-term in equation (9) and the Ω-term in equation (8). In contrast with the Babcock model and its various modifications, in which this “drift” results from the geometry of the differential rotation alone, we here find that the zone of sunspot activity is propagated toward the equator: Eruption of flux at a given latitude serves to strengthen $\vec{B}_s$ and $\vec{B}_q$ at adjacent lower latitudes and to weaken these components at higher latitudes, a process which in turn preferentially enhances the rate of growth of $B_s$ at lower latitudes. As expected, at no time is the field purely poloidal; the epoch at which the local field is purely meridional varies with latitude (Leighton 1964).

The poleward drift of the zero-field line is governed principally by the $T_D$ term of equation (9) in the sense that the rate of approach to the pole depends almost exclusively on the magnitude of the decay time constant $T_D$. The epoch of arrival depends both on $T_D$ and on the details of the flux eruption during the early part of the cycle. This diffusion term also provides a smoothing effect and a coupling between adjacent latitudes.

The amplitude of oscillation is kept finite by the non-linear term in equation (8) and by the “threshold” requirement for eruption. If the oscillation is not in a steady state, these terms will have the effect of lengthening the period for small amplitudes and shortening it for large amplitudes. They also provide coupling between the antisymmetric (dipole) mode and the symmetric (quadrupole) mode, and cause one or the other to die out, depending on the initial relative amplitudes of the two modes and their relative stability.

Equation (10) plays almost no role in the steady state, but it is important in establishing a stable oscillation starting with small initial fields.

b) Fluxropes

Up to this point we have not mentioned the “fluxropes” which were introduced by Babcock and which play a prominent role in his model. Fluxropes are the ropelike, twisted tubes of magnetic flux that might be produced as an initially untwisted fluxtube is rolled along, like a ball bearing, by a radial gradient of the differential rotation field.

The present model does not require the idea of fluxropes, but some qualitative remarks concerning their possible role may be made. The observed polarization effects of sunspot fields (Bumba 1962), the chromospheric “whirls” (Hale 1927), and the marked persistence of large $p$-spots for two, three, or more solar rotations, suggest the reality and importance of twisted fields. On the other hand, Richardson (1941) found that only 30 per cent of the large sunspots that could have shown “whirls” actually did so, and of these a substantial number, perhaps 25 per cent, were in the wrong direction for their hemisphere.

If a twist is produced in the manner assumed by Babcock, then a velocity field in which the lower depths are rotating faster than the surface, and the more equatorial latitudes faster than higher latitudes, will produce a helical twist like the thread of a right-hand screw in the northern hemisphere. When such a tube erupts to produce a sunspot pair, the $p$-spot would apparently show a clockwise whirl, opposite to that generally observed.

An important feature of sunspot eruption which might have its origin in the twist of fluxtubes is the axial tilt of spot groups. This property plays a fundamental role in both Babcock’s model and the present one, but the reason for its existence is not fully understood. However, it is of interest that the stresses in a twisted magnetic field are essentially similar to those in a twisted, stretched rubber band (Alfvén 1950; Dungey 1958). If
a stretched rubber band is twisted to form a right-hand screw and is then allowed to contract slightly until a knot tends to form, it will be noted that this incipient knot, when it is geometrically similar to an upward-rising loop of magnetic field, is also rotated with respect to the band axis in just the sense needed to explain the axial tilt of sunspot groups. A theoretical study of the dynamics of a buoyant, twisted fluxrope might throw light on this effect and also on the eruption time constant $\tau$.

c) Correlation Effects in Sunspot Eruption

We now discuss briefly two features of solar activity which may arise from the mechanisms included in the present model. These are (1) the tendency for new spot groups to appear within the $p$ magnetic region of an older spot group (Becker 1955; Tuominen 1962; Bumba and Howard 1965) and (2) the existence of relatively long-lived longitude zones of enhanced solar activity (Haurwitz 1968, and further references cited therein). We suggest that both of these effects result from the action of the differential rotation field upon the $B_r$ and $B_\theta$ fields, as defined in equation (8). However, we note that, whereas the field components appearing in equation (8) are average fields, so chosen for simplicity in studying the long-term effects, in the actual world we must use equation (2a) with the local, instantaneous fields. An initially random sunspot eruption will thus tend to produce further eruptions: Recall that the $B_\phi$ of a given cycle is produced by the $B_r$ (and $B_\theta$) established by the previous cycle. Thus a $B_r$ having the same sign as that of the preceding cycle will enhance the production of $B_\phi$ of the current cycle, and a $B_\theta$ of opposite sign will suppress it. Now, in the eruption of a sunspot group, the $p$-spots have a $B_r$ of the same sign as the preceding cycle, and the $f$-spots have one of the opposite sign. (This is responsible, as discussed previously, both for the ultimate reversal of the dipole field and for the equatorward propagation of the zone of sunspot eruption.) Therefore, the resulting $p$ and $f$ magnetic regions, as they expand during the next few solar rotations, will, respectively, represent regions of especially high and especially low rates of growth of $B_\phi$. The result is that $B_\phi$ becomes especially strong in the old $p$-region and thereby leads to a greater likelihood of eruption of a new spot group there. This effect works as described only if the radial gradient of rotation rate has the proper sign—with the interior of the Sun rotating faster than the outside.4

The persistence of preferred longitudes of eruption might arise in a similar way: Fluctuations in the rate of sunspot eruptions will sometimes lead to a non-axial net dipole moment of significant strength. Quite large areas might then persistently possess enhanced fields $B_r$ and $B_\theta$ sufficient to increase materially the rate of growth of $B_\phi$, and enhanced sunspot eruption would result. Such long-term persistent regions would tend to occur preferentially near the equator, where the differential rotation would not affect them very strongly. Such long-lived regions are often seen in magnetic synoptic charts (Howard 1968).

The above effects are but two examples of a broad class of correlations that are expected to occur, in which initially random fluctuations are acted upon by the amplification processes to produce correlated phenomena over an extended time period. The present model must be extended to include variations of field strengths with longitude in order fully to analyze this aspect of solar activity. However, the possibility—indeed, the strong likelihood—of such correlative effects is clear, and suggests that caution be used in inferring the existence of intrinsic asymmetries in the Sun on the basis of statistical analyses of solar phenomena such as sunspots, flares, and the like. Frames of apparent rigid-body rotation having synodic periods near 274 have been inferred from analyses of the longitudes of occurrence of proton flares, polar cap absorptions, sunspot numbers, etc. (Losh 1938; Warwick 1965; Haurwitz 1968; see Haurwitz 1968 for further references). In each case a clustering of the events of interest toward certain "preferred" longitudes

4 In the opposite case, the butterfly diagram progresses toward the pole, and new spot groups would tend to appear in the $f$ magnetic regions.
is found, and is shown to be highly significant when tested against the hypothesis of independent, random longitudes of occurrence for the events. I suggest that the high confidence with which this hypothesis may be rejected is due, not to an underlying rigid-body frame, but to an unexpectedly strong correlation in longitude of occurrence of supposedly independent events.

d) Sunspot Fields and Total Erupted Flux

If the value of the thickness $H$ of the shear layer were known, one could estimate the field strengths in sunspots on the basis of the model. As a rough guess, values in the range $0.05R < H < 0.2R$ seem appropriate: Smaller values of $H$ would pose questions concerning the means by which the submerged field could resist being carried to the surface by the supergranulation circulation, while larger values would exceed the estimated thickness of the convection zone itself. Taking $H = 0.1R$ as a basis for discussion, one finds from Table 1 that typical values of $B_r$ and $B_{\phi m}$ would be 200–300 gauss and 1000–2000 gauss, respectively. The agreement with the range of field strengths found in sunspots seems quite satisfactory, especially if sunspots were to arise preferentially from local concentrations of field.

One may also estimate the total erupted flux during a solar cycle. If the maximum radial field at the pole is taken as 1 gauss, as in Table 1, a summation of all flux erupted in one "wing" of an average "butterfly" is found to be about 100R$^2$, gauss cm$^2$ or about $0.5 \times 10^{24}$ Mx. Grotian and Künzel (1950) found that the average yearly flux associated with $p$-spots in one solar hemisphere near sunspot maximum was about $2 \times 10^{22}$ Mx. Allowing perhaps a factor of 2 for unseen flux near the spots and a similar factor for unseen spots, such a flux occurring for 4 years near maximum would give $0.3 \times 10^{24}$ Mx over a cycle. The conclusion is that if the model is normalized to a field amplitude which matches observed polar field strengths, the total erupted flux matches observed total fluxes reasonably well.

VIII. CONCLUSION

In view of the rather broad quantitative agreement of the present model with the known features of the solar cycle, it seems reasonable to conclude that the mechanism of field amplification by the solar differential rotation is basic to the operation of the cycle, and that other factors which have been suggested as contributing to the solar cycle, such as meridional circulatory currents and torsional oscillations, are unlikely to be important. Special interest is thus focused on theoretical studies of the radial and latitudinal variation of the differential rotation, the detailed nature of the subsurface field and of the fluctuations which govern sunspot eruption, and the dynamics of the eruption process itself.

In his paper on the topology of the Sun's magnetic field, Babcock stressed the need for "more and better measurements of the distribution and quantity of magnetic flux in BMR's as a function of age." The results found here only emphasize that a comprehensive test of any model of the solar cycle will ultimately require a knowledge of the strength, spatial distribution, and history of the solar magnetic field spanning a considerable time period. The existing indices of solar activity, while forming a valuable continuous record covering several complete cycles, will probably be of diminishing utility unless they can be reliably related to physically significant quantities. It is also important to realize that certain properties of sunspots may be functions of both latitude and time, i.e., of position in the butterfly diagram: axial tilts, sizes, and correlated series of spot groups are possible examples. New analyses of existing sunspot data with such ideas in mind might prove quite fruitful.

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